

# Self Similarities of the Tower of Hanoi Graphs and a proof of the Frame-Stewart Conjecture

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**Abstract.** Considering the symmetries and self similarity properties of the corresponding labeled graphs, it is shown that the minimal number of moves in the Tower of Hanoi game with  $p = 4$  pegs and  $n \geq p$  disks satisfies the recursive formula  $F(p, n) = \min_{1 \leq i \leq n-1} \{2F(p, i) + F(p-1, n-i)\}$  which proves the strong Frame-Stewart conjecture for the case  $p = 4$ . The method can be generalized to  $p > 4$ .

**1. INTRODUCTION.** The Tower of Hanoi game with  $p \geq 3$  pegs and  $n \geq 0$  disks is a popular game in which disks have to be moved from one to another peg, obeying the rule that a larger disk can never be put onto a smaller one [3]. The case  $p = 4$  is sometimes called Reve's puzzle. It is usual to study the game by regarding the graph of possible positions and legal moves among them. The most prominent open problem is the Frame-Stewart conjecture about the minimality of a certain algorithm for the distance between so called perfect states. The known solution to the Problem 3918 [5] that has been reinvented several times (see [3] for historical facts) appears to be a very natural one, but the proof of optimality in general case is not known until today. The two solutions proposed by Frame and Stewart [2, 6] are known to be equivalent [4] and the number of steps needed for  $p$  pegs and  $n \geq p$  disks is given by the recursive formula

$$F(p, n) = \min_{1 \leq i \leq n-1} \{2F(p, i) + F(p-1, n-i)\}. \quad (1)$$

As it is trivial that  $F(p, n) = 2n - 1$  for  $n < p$ , and it is well-known that  $F(3, n) = 2^n - 1$ , formula (1) determines  $F(p, n)$  for all  $p \geq 3$  and  $n \geq 0$ . The literature on the topic is enormous, for example there are 352 references in [3]. Very recently, a proof of optimality for the case  $p = 4$  appeared in [1].

Here we first observe some self similarity properties of the labeled graphs that are isomorphic to Hanoi graphs. The symmetries can rather naturally be observed from drawings that, to our surprise, are used very rarely. In fact, in an attempt to Google search for drawings of Hanoi graphs, the present author found a similar drawing in only one paper [7], which seems to be unpublished students' homework(!). The structure of the Hanoi graphs allows to prove a lemma about existence of shortest paths of certain structure that in turn provides a proof that the Frame-Stewart algorithm is optimal. This solves the famous Frame-Stewart conjecture.

The rules of the game are simple. There are  $p$  pegs and  $n$  disks, that all differ in size (diameter). In the beginning, all the disks are at one peg, (legally) ordered by size. It is not allowed to put a larger disk onto a smaller one at any time. In one move, a disk

that is on the top at one peg is put to the top at another peg. The task is to reach a state in which all the disks from original peg are at another peg so that the number of moves is minimal.

Only basic notions of graph theory will be used. A graph is a pair of sets  $G = (V(G), E(G))$ , where  $V(G)$  is an arbitrary set of vertices, and  $E(G)$  is a set of pairs of vertices. Usual notation is  $e = uv$  meaning that edge  $e$  connects vertices  $u$  and  $v$ . We work with labeled graphs, i.e. graphs with a labeling function that assigns a label to each vertex. Two (unlabeled) graphs  $G$  and  $H$  are isomorphic when there is a bijection (called isomorphism)  $\alpha : V(G) \rightarrow V(H)$  such that  $u$  and  $v$  are connected in  $G$  if and only if  $\alpha(u)$  and  $\alpha(v)$  are connected in  $H$ . A walk is a sequence of vertices and edges  $v_0 e_1 v_1 e_2 v_2 \dots e_k v_k$  such that  $e_i = v_{i-1} v_i$ . A walk is also determined either by the sequence of vertices  $v_0 v_1 v_2 \dots v_k$  or by the sequence of edges  $e_1 e_2 \dots e_k$ . The length of a walk is the number of edges on it. A path is a walk in which all vertices are distinct. The distance between two vertices is the length of a shortest path. A subgraph  $H$  of a graph  $G$  is isometric subgraph, if the distance between any two vertices in  $H$  is equal to the distance in  $G$ . For notions not recalled here see, for example [8].

The rest of the paper is organized as follows. In the next section, we construct labeled graphs  $G_n^{(p)}$  corresponding to the Tower of Hanoi game with  $p$  pegs and  $n$  disks. In Section 3, a lemma about the self similarity of these graphs is proved and some related facts are given. Sections 4 and 5 provide proof of existence of shortest paths of certain form which in turn gives a lower bound on length of the shortest paths between certain vertices. This implies the main result, a proof of the Frame-Stewart conjecture for four pegs that appears in Section 6.

**2. THE CONSTRUCTION OF LABELED GRAPHS  $G_N^{(P)}$ .** Before giving the general definition, we start with the  $p = 4$  pegs example. Labels of the graphs  $G_n^{(4)}$  will be words of length  $n$  over the four letter alphabet  $\mathcal{A}_4 = \{A, B, C, D\}$ .

Let  $G_1^{(4)}$  be the tetrahedron graph (complete graph on 4 vertices), and the vertices labeled with  $A, B, C$ , and  $D$ .

Given  $G_n^{(4)}$ , we construct  $G_{n+1}^{(4)}$  as follows. Take four copies of  $G_n^{(4)}$ : in the first copy, denoted by  $AG_n^{(4)}$  replace each label  $*$  with label  $A*$  (i.e. add an  $A$  at the beginning of each label). Similarly, in  $BG_n^{(4)}$  replace each label  $*$  with label  $B*$ , in  $CG_n^{(4)}$  replace each label  $*$  with label  $C*$ , and in  $DG_n^{(4)}$  replace each label  $*$  with label  $D*$ .

Finally connect some pairs of vertices from different copies of  $G_n^{(4)}$  with edges by the following rule. Let  $X, Y \in \mathcal{A}_4$ ,  $X \neq Y$ . Two vertices  $u \in V(XG_n^{(4)})$  and  $v \in V(YG_n^{(4)})$  are connected if and only if the labels of  $u$  and  $v$  without the first letter are equal and are words over  $\mathcal{A}_4 - \{X, Y\}$ . (i.e. the labels do not contain any  $X$  or  $Y$ ). The graphs  $G_1^{(4)}$  and  $G_2^{(4)}$  are drawn on Fig. 1 and the graph  $G_3^{(4)}$  is on Fig. 2.

In other words, from the definition it follows that two vertices from different copies of  $G_n^{(4)}$  are connected in  $G_{n+1}^{(4)}$  exactly when their labels are equal (in the last  $n$  letters) and the labels are words over an alphabet of two letters.

**Remark.** The graph  $G_n^{(4)}$  is isomorphic to the Hanoi graph of the game with 4 pegs and  $n$  disks. Just interpret the labels naturally as:  $i$ -th letter in the label is the position of the  $i$ -th disk. For example, the first letter gives the position of the largest disk.

We now turn to general definition, for arbitrary  $p \geq 3$ . A generalization to  $G_n^{(p)}$  is straightforward :

**Definition.** Let  $G_1^{(p)}$  be a complete graph with  $p$  distinct labels on vertices, say using letters from alphabet  $\mathcal{A}_p$ . Construct  $G_{n+1}^{(p)}$  using  $p$  copies of  $G_n^{(p)}$  as above, i.e. in each copy of  $G_n^{(p)}$  use a different letter as a prefix for labels. As before, connect two vertices

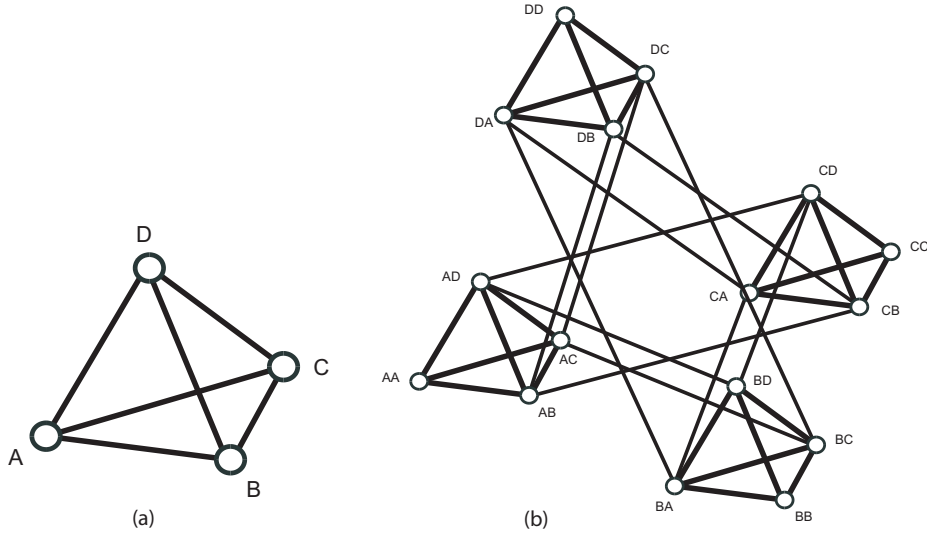


Figure 1. The graphs  $G_1^{(4)}$  and  $G_2^{(4)}$ .

$u \in V(XG_n^{(p)})$  and  $v \in V(YG_n^{(p)})$  if and only if the labels of  $u$  and  $v$  without the first letter are equal and are words over  $\mathcal{A}_p - \{X, Y\}$  (i.e. the labels do not contain any  $X$  or  $Y$ ).

**Remark.** By construction, the graph  $G_n^{(p)}$  is the graph of the Tower of Hanoi game with  $k$  pegs and  $n$  disks. (As unlabeled graph, it is therefore isomorphic to the Hanoi graph,  $H_n^p$  in notation of [3].) The Frame-Stewart conjecture says that the minimal number of moves between two perfect states is determined by recursion (1) which is equivalent to statement that this is the length of a shortest path in  $G_n^{(p)}$  between two perfect vertices, for example vertices with labels  $A^n = AA\dots A$  and  $B^n = BB\dots B$ .

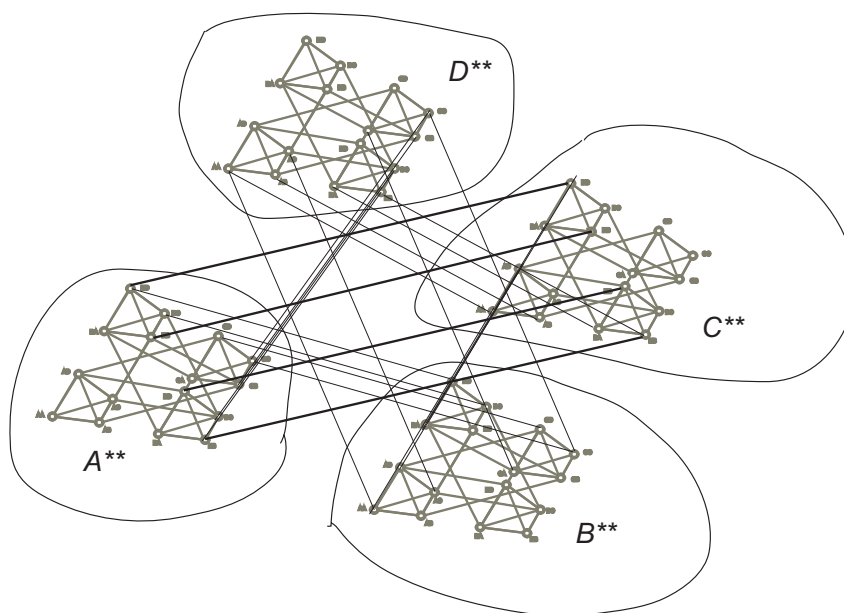
**3. SELF SIMILARITY.** It is well known that the Hanoi graphs are highly symmetric. Here we first emphasize one of the properties that motivated the idea used in the argument given below. Some more properties are listed below for later reference. We do not give detailed proofs as we believe that these results are not new, and are recalled here for completeness of presentation.

Given  $G_n^{(p)}$  and  $1 \leq i \leq n$  define the equivalence relation  $R_i$  on the set of vertices  $V(G_n^{(p)})$  as follows: two vertices are equivalent when their labels coincide in the first  $n - i$  letters. Define the graph  $G_n^{(p)} / R_i$  on equivalence classes (as vertices) by connecting two equivalence classes if there is an edge  $G_n^{(p)}$  that connects a pair of vertices from the two classes. The definition directly implies:

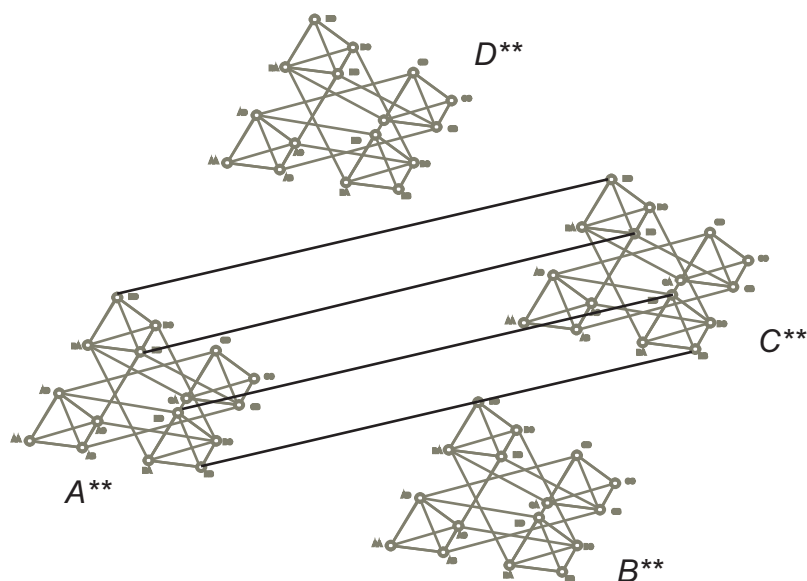
**Lemma 1.**  $G_n^{(p)} / R_i$  is isomorphic to  $G_{n-i}^{(p)}$ .

Having in mind this structure, we will (for fixed  $i$ ) make a distinction between the edges within equivalence classes and the edges connecting different classes. The later will be called *bridges* and the edges within equivalence classes will be referred to as *local edges*. Thus bridges correspond to moves of the largest  $n - i$  disks and local edges to moves of the smallest  $i$  disks. (See Fig. 2 and Fig. 3, where the bridges between classes  $A^{**}$  and  $C^{**}$  in  $G_3^{(4)}$  are shown.)

By definition of  $R_i$ , each equivalence class of  $R_i$  induces a subgraph of  $G_n^{(p)}$  that is isomorphic to  $G_i^{(p)}$ . Furthermore, recall that any bridge connects two vertices (from



**Figure 2.** The graph  $G_3^{(4)}$ ; with the four equivalence classes ( $i = 1$ ).



**Figure 3.** The four equivalence classes ( $i = 1$ ) of the graph  $G_3^{(4)}$  and all edges (bridges) between two classes.

different equivalence classes) with the same labels (i.e. with labels that match in the last  $i$  letters).

We state two more properties for a later reference. The proofs follow directly from the definitions. Maybe even more natural argument is to consider the meaning in terms of the game, namely (1) fixing positions of the largest  $n - i$  disks clearly results in a game with  $i$  disks and (2) putting the smallest  $i$  disks to one of the pegs forbids the

moves to that peg, hence exactly  $p - 1$  pegs are free to move the largest  $n - i$  disks at.

**Lemma 2.** *Let  $W$  be a arbitrary word of  $n - i$  letters over alphabet  $\mathcal{A}_p$ . Then  $WG_i^{(p)}$  are isometric subgraphs of  $G_n^{(p)}$ .*

**Lemma 3.** *The vertices  $WG_i^{(p)}$  of  $G_n^{(p)} / R_i$  where  $W$  is an arbitrary word of  $n - i$  letters over alphabet  $\mathcal{A}_p - X$  induce a subgraph that is isomorphic to  $G_{n-i}^{(p-1)}$ .*

We conclude the section with a couple of facts that will be useful later. As above, the proofs are not difficult, for example by considering the meaning of the distances in terms of the game. In short, the first claim follows by obvious symmetry (just replace the role of  $A$  and  $B$ ). The second claim is obvious because if we can solve the bigger task, then we also can solve the easier task (and need not move the largest disk). Details are left to the reader.

**Fact 1.** *Let  $A$  and  $B$  be arbitrary letters from alphabet  $\mathcal{A}_p$ . Let  $W$  be a word of  $i$  letters over alphabet  $\mathcal{A}_p - \{A, B\}$ . Then the distance between vertices with labels  $A^i$  and  $W$  is equal to the distance between vertices with labels  $B^i$  and  $W$ . Hence, there is no shortest path connecting vertices with labels  $A^i$  and  $W$  that meets  $B^i$  in  $G_i^p$ .*

**Fact 2.** *Let  $W$  be a word of  $i$  letters over alphabet  $\mathcal{A}_p - \{A\}$ . Then there exists  $C \neq A$  such that the distance between vertices with labels  $C^i$  and  $W$  is strictly smaller than the distance between vertices with labels  $A^i$  and  $W$ .*

**Remark.** Idea of proof of Fact 2 : On any path  $P$  from  $W$  (i.e. vertex with label  $W$ ) to  $A^i$ , the largest disk will be moved to peg  $A$ , so the path  $P$  can be written as  $W \rightarrow W^* \rightarrow AW^{**} \rightarrow A^i$ , where  $W^*$  and  $AW^{**}$  are two labels that differ only in the first letter. Let the first letter of  $W^*$  be  $C$ . Then there is a path  $P^*$  from  $W$  to  $C^i$ , i.e.  $W \rightarrow W^* = CW^{**} \rightarrow C^i$ , and  $P^*$  is shorter than  $P$ .

**4. SHORTEST PATHS.** Let  $n \geq 2$ , as the case  $n = 1$  is trivial. Let  $\mathcal{A}_p$  be an alphabet of  $p$  letters, and  $A, B \in \mathcal{A}_p$ .

Let  $P$  be a shortest path from vertex  $a$  with label  $A^n$  to vertex  $b$  with label  $B^n$ . Below, in Proposition 1 we will assume that on the path  $P$  there is a vertex that has a label of the form  $A^{n-i}X^i$ , where  $X \in \mathcal{A}_p - \{A, B\}$  (i.e.  $X$  is not  $A$  nor  $B$ ). We call such a vertex *special*. It may seem obvious that there is a special vertex on every shortest path from  $a$  and  $b$ . However, this is not the case as pointed out by Ciril Petr and Sandi Klavžar. A slightly weaker, but still sufficient, statement can be proved

**Lemma 4.** *Let  $p = 4$ . There is a shortest path with at least one special vertex on every path between vertices  $a$  and  $b$  with labels  $A^n$  and  $B^n$ .*

The proof of Lemma is postponed to the next section. It is based on induction, in which the small cases are, due to large number of them, rather tedious task. Alternatively, it can be checked by computer using a straightforward application of a shortest path algorithm. We believe that the same technique can be used for  $p > 5$  and conjecture

**Conjecture 1.** *There is a shortest path with at least one special vertex on every path between vertices  $a$  and  $b$  with labels  $A^n$  and  $B^n$ .*

Now we will show that there is a shortest path with certain structure. More precisely,

**Proposition 1.** *Let  $P$  be a shortest path connecting vertices  $a$  and  $b$  with labels  $A^n$  and  $B^n$  and let  $1 \leq i \leq n$ . If there is a special vertex on  $P$  with label of the form  $A^{n-i}X^i$ , where  $X \in \mathcal{A}_p - \{A, B\}$  then there is a shortest path  $Q$  from  $a$  to  $b$  in  $G_n^{(p)}$ , which is a concatenation of three subpaths  $Q_1, Q_2, Q_3$  such that  $Q_1$  and  $Q_3$  only use local edges and  $Q_2$  only uses bridges.*

*Proof.* Consider a shortest path  $P$  from  $a$  to  $b$  (vertices with labels  $A^n$  and  $B^n$ ). Let  $i = i(P)$  be maximal with property that there is a special vertex  $s = s(i)$  with label  $A^{n-i}X^i$  on the path  $P$ .

Denote by  $P_1$  the first part of  $P$ , from  $a$  to  $s$ , and observe that because of Lemma 2 we can, without loss of generality, assume that there are only local edges on  $P_1$ . As  $P$  is a shortest path,  $Q_1 = P_1$  must be a shortest path from  $a$  to  $s$  within the subgraph  $A^{n-i}G_i^{(p)}$ .

On path  $P$  there must be at least one bridge after  $P$  meets  $s$  because labels  $A^n$  and  $B^n$  differ in the first  $n - i \geq 1$  letters that can only be changed when traversing bridges. First we show that there is a shortest path such that the edge used after visiting  $s$  is a bridge:

**Claim 1.** *Let  $P$  be a shortest path from  $a$  to  $b$  (with labels  $A^n$  and  $B^n$ ) that meets vertex  $s = s(i)$  with label  $A^{n-i}X^i$ ,  $i = i(P)$ . Then there is a path  $P'$  of the same length such that the first edge after visiting  $s$  is a bridge.*

**Proof (of the Claim).** If  $P$  has the property claimed then  $P' = P$  and we are done. Now assume that  $P = P_1P_2fP_3$  where  $P_1$  is a shortest path from  $a$  to  $s$ ,  $P_2$  is a subpath of local edges from  $s$  to  $w$ ,  $f$  is a bridge, and  $P_3$  is the rest of  $P$ .

We distinguish two cases. First, let  $w$  be a vertex with label  $A^{n-i}W$  where  $W$  is a word using letter  $X$ . This implies that  $f$  is a bridge that moves a disk from peg  $A$  to a peg that is not peg  $X$ , say  $f$  moves  $n - i$ -th disk from  $A$  to  $Y \neq X$ . This implies that we can replace subpath  $P_2f$  with  $f'P'_2$ , where  $f'$  is the edge connecting vertex  $s$  (with label  $A^{n-i}X^i$ ) and vertex  $s'$  that has label  $A^{n-i-1}YX^i$  and  $P'_2$  is a copy of  $P_2$  in the subgraph  $A^{n-i-1}YG_i^{(p)}$ . (Formally, the path  $P'_2$  is constructed from  $P_2$  by replacing every vertex on  $P_2$  (with label  $A^{n-i}*$ ) with the vertex with label  $A^{n-i-1}Y*$ .)

Now assume that  $w$  is a vertex with label  $A^{n-i}W$  where  $W$  is a word without letter  $X$ . In this case the subpath  $P_1P_2$  (within  $A^{n-i}G_i^{(p)}$ ) starts at  $a$  (with label  $A^n$ ), visits first vertex with label  $A^{n-i}X^i$  and then reaches  $w$ . However, there is a strictly shorter path from  $a$  to  $w$  which contradicts the assumption that  $P$  is a shortest path (recall Fact 1). ■(Claim)

Now we will prove that there is no need to use local edges between bridges, i.e. that there is a shortest path which, after visiting  $s$ , first traverses all bridges, and then traverses only local edges. The next claim shows that we can always conveniently change position of the next displaced bridge. More precisely,

**Claim 2.** *Let  $P$  be a shortest path of the form  $P = P_1P_2LfP_3$ , where  $P_1$  is a shortest path from  $a$  to  $s$ ,  $P_2$  is a path of bridges from  $s$  to  $t$ ,  $L$  is a local path,  $f$  is a bridge, and  $P_3$  is the rest of  $P$ . Then there is shortest path  $P' = P_1P_2f'L'P_3$ , where  $f'$  is a bridge and  $L'$  is a local path of the same length as  $L$ .*

**Proof (of the Claim).** The local path  $L$  is a path from vertex  $t$  to a vertex, say  $w$ . Let the label of  $t$  be  $ZX^i$  where  $Z$  is a word of length  $n - i$  that does not contain any  $X$ . Then the label of  $w$  must be of the form  $ZW$ .

We distinguish two cases. First, assume the word  $W$  contains at least one letter  $X$ . Therefore bridge  $f$  that moves one of the large disks may not move a disk to peg  $X$ . (More precisely, edge  $f$  connects  $w$  and  $u$ , and the label of  $u$  must be  $Z'W$ , where the



words  $Z'$  and  $Z$  differ in one letter at one position, and neither  $Z'$  nor  $Z$  contain any letter  $X$ .)

Hence we can define  $f'$  to be the bridge that connects  $t$  (with label  $ZX^i$ ) and the vertex with label  $Z'X^i$ , denote it  $r$ . Furthermore, let  $L'$  be a copy of  $L$  in  $Z'G_i^{(p)}$  that connects vertices  $r$  and  $u$ . By construction, paths  $f'L'$  and  $Lf$  both connect  $t$  and  $u$  and are of the same length, as needed.

Now assume that  $w$  is a vertex with label  $ZW$  where  $W$  is a word without letter  $X$ . In this case, we can construct a shorter path from  $a$  to  $w$ , which contradicts the assumption that  $P$  is a shortest path. The argument is as follows.

First, observe that, because there is no  $X$  in  $W$ , the first letter of  $W$  is  $Y \neq X$ . (In other words, the  $i$ -th smallest disk is on peg  $Y$  that is not peg  $X$ .)

Second, by symmetry, existence of the path  $P_2$  (connecting vertices with labels  $A^{n-i}X^i$  and  $ZX^i$ ) implies the existence of a path  $P'_2$  of the same length that connects vertices with labels  $A^{n-i}Y^i$  and  $ZY^i$ . (Just replace any occurrence of  $Y$  in  $P_2$  with  $X$  (and vice versa) to get  $P'_2$ .)

Furthermore, by symmetry, there is a path  $P'_1$  (of the same length as  $P_1$  that connects vertex  $a$  and the vertex with label  $A^{n-i}Y^i$ ).

Finally, recalling Fact 2, in  $ZG_i^{(p)}$  there is a path  $L'$  from the vertex with label  $ZYY^{i-1}$  to vertex  $w$  that is strictly shorter than  $L$ .

Summarizing, the path  $P'_1P'_2L'$  from  $a$  to  $w$  is shorter than  $P_1P_2L$ , contradicting the minimality of  $P$ . ■(Claim)

By inductive application of the last Claim we prove that any shortest path  $P$  can be replaced by a shortest path  $Q = Q_1Q_2Q_3$  where  $Q_2$  is a path of bridges while  $Q_1$  and  $Q_3$  are path of local edges. ■

**5. SPECIAL VERTICES - PARTIAL PROOF OF CONJECTURE ??.** In this section we will prove Lemma 4. The proof is by induction. We emphasize that the inductive step is proved for general  $p$  while we are only able to prove the base step(s) for each  $p$  separately.

We begin by introducing some more notation. In the graph  $G_{n+1}^{(p)}$ , we consider shortest paths from  $a$  to the set of vertices on the *border* of  $AG_n^{(p)}$ , i.e. to vertices from which there are edges going out to other subgraphs  $XG_n^{(p)}$ ,  $X \neq A$ . The border  $S_n$  is, by definition of  $G_{n+1}^{(p)}$ , a union of  $S_n^{AX}$ , where  $S_n^{AX}$  is a set of vertices with labels that do not use letters  $A$  and  $X$ :  $S_n = \bigcup_{X \in A-A} S_n^{AX}$ . As  $AG_n^{(p)}$  is just a copy of  $G_n^{(p)}$

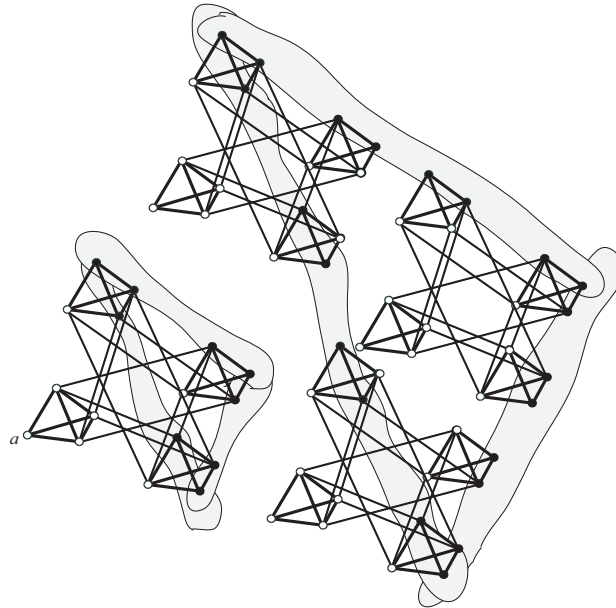
we can recursively define  $S_{n-1}$  in  $G_n^{(p)}$ , and because  $AG_n^{(p)}$  is a subgraph of  $G_{n+1}^{(p)}$ , this also gives a definition of  $S_{n-1}$  in  $G_{n+1}^{(p)}$ . Thus we define the sets  $S_i$  in  $G_{n+1}^{(p)}$  for  $i = 1, 2, \dots, n$  (see Figure 4). Obviously,

**Fact 3.** Any shortest path from  $a$  to  $v \in S_n$  meets  $S_{n-1}$ .

As already mentioned, there are examples, in which there is a unique shortest path from  $a$  to some vertices on the border. We conjecture that this is only possible for small  $n$ .

**Conjecture 2.** For any  $p \geq 3$  there is  $n_0 \geq 2$  such that for any  $v \in S_{n_0}$  there is a shortest path from  $a$  to  $v$  that meets a special vertex.

For  $p = 4$ , the validity of conjecture can be proved by regarding  $S_6$  in the graph  $G_7^{(4)}$ . In fact the present author performed a tedious case analysis "by hand" using a tool for drawing shortest paths in Hanoi graphs by Igor Pesek. As a curiosity, let us



**Figure 4.** Subgraph  $A^{n-3}G_3^{(4)}$  with emphasized vertex sets  $S_2$  in  $S_3$ . Edges corresponding to the moves of the 3rd disk are not drawn.

mention that the only vertices on the boundary  $S_5$  that have a unique shortest path to  $a$  are the six vertices with labels  $A...ACCD CD$ ,  $A...ADD CD$ , (and, by symmetry,  $A...ACC BCB$ ,  $A...ABBC BC$ ,  $A...ABB DBD$ ,  $A...ADDB DB$ ), two of them passing the vertex with label  $A...AAAA AB$ . On  $S_6$ , there is no vertex with a unique shortest path to vertex  $a$ , and in particular, in each case at least one of the paths avoids vertex with label  $A...AAAA AB$ .

We thus know that

**Fact 4.** For any  $v \in S_6$  in the graph  $G_7^{(4)}$ , there is a shortest path from  $a$  to  $v$  that meets a special vertex.

Hence, by induction step (Fact 3), for any  $v \in S_n$  in the graph  $G_{n+1}^{(4)}$  there is a shortest path from  $a$  to  $v$  that meets a special vertex, for any  $n \geq 7$ .

Recall that any shortest path between  $a$  and  $b$  (with labels  $A^n$  and  $B^n$ ) in  $G_n^{(p)}$  meets  $S_{n-1}$ , more precisely  $S_{n-1}^{AB}$ . Provided validity of Conjecture 2 (that is proved above for case  $p = 4$ ), we have the existence of a shortest path between  $a$  and  $b$  (with labels  $A^n$  and  $B^n$ ) that meets a special vertex. Furthermore, for  $p = 4$  and  $n < 7$  it is straightforward to construct shortest paths that meet a special vertex. This concludes the proof of Lemma 4.

**Remark.** Clearly, the cases  $p > 4$  can be handled along the same lines. The size of graphs however is probably too large to check without computer assistance.

**6. THE MAIN RESULT.** Proposition 1 implies that there is a shortest path between vertices  $a$  and  $b$  (with labels  $A^n$  and  $B^n$ ) in which bridges are all sandwiched together between two local paths. Hence, it suffices to consider the paths of this form.

**Lemma 5.** Assume validity of Conjecture 1. Let  $P$  be a path connecting vertices with labels  $A^n$  and  $B^n$ , and with a special vertex on  $P$  with label of the form  $A^{n-i}X^i$ , where  $X \in \mathcal{A} - \{A, B\}$ . Assume  $P$  is a concatenation of three subpaths  $P_1$ ,  $P_2$ , and



$P_3$  such that  $P_1$  and  $P_3$  only use local edges and  $P_2$  only uses bridges. Then the length of  $P$  is  $F(p, i) + F(p - 1, n - i) + F(p, i)$ .

*Proof.* Let the special vertex  $s$  on  $P$  be of the form  $A^{n-i}X^i$ . Then the last  $i$  letters of labels of all the vertices on  $P_2$  are  $X^i$ , which implies that  $P_2$  never meets the copies  $WG_i^{(p)}$  where  $X$  appears in the word  $W$ . Recall that by Lemma 3, the subgraph induced on vertices  $WG_{n-i}^{(p)}$  where  $W$  is a word of length  $n - i$  over alphabet  $\mathcal{A}_p - X$  is isomorphic to  $G_{n-i}^{(p-1)}$ .

The lengths of subpaths  $P_1, P_2, P_3$  are thus bounded by  $F(p, i), F(p - 1, n - i)$ , and  $F(p, i)$ , respectively. Hence the statement of the Lemma follows. ■

**Theorem 1.** Assume validity of Conjecture 1. The distance between vertices with labels  $A^n$  and  $B^n$  in  $G_n^{(p)}$  is  $\min_{1 \leq i \leq n} \{2F(p, i) + F(p - 1, n - i)\}$ .

*Proof.* Consider a shortest path  $P$  between vertices with labels  $A^n$  and  $B^n$ . There must be a special vertex on  $P$ , and, by Lemma 1, a shortest path  $Q$  for which the length is given by Lemma 5. Recall [4, 3] that there are algorithms for which the number of moves is given by Eq. (1). Therefore the length of a shortest path  $P$  is of the form  $2F(p, i(P)) + F(p - 1, n - i(P))$ , as claimed. ■

Theorem 1 and Lemma 4 imply

**Theorem 2.** The strong Frame-Stewart conjecture is true for  $p = 4$ .

Recall that for any  $p$ , validity of Conjecture 1 implies the strong Frame-Stewart conjecture for that  $p$ .

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